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# On the Stability of Interconnected Systems*

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*Theorems are presented concerning conditions for the input-output stability of interconnected dynamical systems. Results in the area of input-output stability are often partitioned into two categories: small-gain type-results and passivity-type results. The main theorem given here does not fall into either of these categories, but is most closely related to the passivity-type results. The theorem involves a new class of interconnection operators that is a substantial generalization of the familiar set of nonnegative operators defined on a space of vector-valued functions.*

## I. INTRODUCTION

In this paper, theorems are presented concerning the input-output stability of interconnected systems.[†] Results in the area of input-output stability are often partitioned into two categories: small-gain type results and passivity-type results. The main theorem given here does not fall into either of these categories, but is most closely related to the passivity-type results. The theorem involves a new class of interconnection operators that is a substantial generalization of the familiar set of nonnegative operators defined on a space of vector-valued functions.

The mathematical model considered throughout the paper is described in Section II, and results of a general nature concerning the model are given in Section III. The case in which the interconnection operator has a certain matrix representation is treated in considerable detail in Section IV. In Section 4.5, a specific example is given of a stable interconnected system for which the interconnection matrix does not meet the nonnegative-definiteness requirement of the criterion given in Ref. 7, which contains the most pertinent earlier stability result.

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* This paper was presented at the 1978 IEEE Symposium on Circuits and Systems (New York, May 17–19, 1978).

† For background material in book form concerning intput-output stability, see, for example, Refs. 1–4. Interconnected systems (which are systems whose natural or artificial decomposition into subsystems plays a prominent role in their mathematical analysis) have been considered by many researchers. See, for example, Refs. 3, 5, 6, and 7. Although some interesting and significant results have been obtained concerning the stability of interconnected systems, the theory is very much in its initial stages of development.

The main purpose of the paper is to introduce a new concept that is believed to be useful. No attempt is made to present the sharpest possible stability results that the concept can be used to obtain.

## II. THE MODEL

### 2.1 Preliminaries

Let $K$ denote a real linear space that contains a normed linear inner-product space $L$ with inner product $(\cdot,\cdot)$ and norm $|\cdot|$ related by $|f| = (f,f)^{1/2}$ for $f \in L$. (Of particular interest to us is the case in which $L$ is the set $L_2$ of all real Lebesgue square-integrable functions defined on the half line $[0,\infty)$ with the usual inner product, and $K$ is the "extended" set $E_2$ of real functions defined on $[0,\infty)$ such that each function is square integrable on $[0,\tau]$ for any nonnegative number $\tau$.)

For each $\tau \geq 0$, let $P_\tau$ denote a linear mapping of $K$ into $L$ (e.g., if $K = E_2$, let $P_\tau$ be defined by $(P_\tau f)(t) = f(t)$ for $t \in [0,\tau]$ and $f(t) = 0$ for $t > \tau$, where $f$ is an arbitrary element of $E_2$).

Let $K$, $L$, and $P_\tau$ be such that $(i)$ $g \in L$ if and only if $g \in K$ and $\sup_\tau |P_\tau g| < \infty$, $(ii)$ $|g| = \sup_\tau |P_\tau g|$ for $g \in L$, and $(iii)$ $(P_\tau f,g) = (P_\tau f,P_\tau g)$ and $|P_\tau f| \leq |f|$ for $f$ and $g$ in $L$ and $\tau \geq 0$.

We let $L^n$ and $K^n$, in which $n$ is any positive integer, denote the $n$-fold Cartesian product of $L$ and $K$, respectively. The norm of an element $h = (h_1,h_2,\ldots,h_n)$ of $L^n$ is denoted by $|h|$ and is defined by $|h| = (\Sigma_i |h_i|^2)^{1/2}$.

It is assumed that $L$ contains $n$ elements $e_1,e_2,\ldots,e_n$ such that $|e_i| = 1$ for each $i$ and $(e_i,e_j) = 0$ for $i \neq j$.*

We say that an operator $T$ that maps $K$ into itself (i.e., an operator $T$ in $K$) is *causal* if and only if $P_\tau T = P_\tau T P_\tau$ on $K$ for all $\tau \geq 0$.

### 2.2 The basic equations

Throughout the paper, attention is focused on an interconnected system governed by

$$x_i + \sum_{j=1}^{n} A_{ij}B_j x_j = y_i, \quad i = 1,2,\ldots,n, \tag{1}$$

in which $(A.1)$: $x_i$ and $y_i$ belong to $K$ for all $i$, and $A_{ij}$ and $B_j$ are causal operators in $K$ for all $i$ and $j$.

In (1), each $B_j$ is associated (sometimes somewhat indirectly) with a subsystem, and the $A_{ij}$ ordinarily take into account the way in which the subsystems interact. Typically, it is not difficult to show the existence of a solution $x_1, x_2, \ldots,x_n$ of (1) for any given $y_1, y_2, \ldots,y_n$ under some weak additional hypotheses. (Successive-approximation type arguments of the kind commonly used in connection with nonlinear Volterra integral equations often suffice.)

* This assumption is used only in Section IV.

We assume that (*A.2*): each $B_j$ is nonnegative in $L$, in the sense that each $B_j$ maps $L$ into $L$ and there exists a nonnegative constant $\alpha$ such that

$$(B_j w, w) \geq \alpha |w|^2 \tag{2}$$

for $w \in L$ and all $j$. It is assumed also that (*A.3*): each $A_{ij}$ maps $L$ into itself, and there is a positive constant $\gamma$ such that

$$|A_{ij} w| \leq \gamma |w| \tag{3}$$

for $w \in L$ and all $i$ and $j$.*

It is often convenient to write (1) in the form

$$x + ABx = y, \tag{4}$$

in which $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, and $A$ and $B$ are the mappings of $K^n$ into $K^n$ defined by $(Af)_i = \Sigma_j A_{ij} f_j$ and $(Bf)_i = B_i f_i$ for all $f \in K^n$ and each $i$.

### 2.3 Definition of stability

We say that (4) is *L-stable* if and only if $y \in L^n$ implies that $x \in L^n$ with $|x| \leq \rho(|y|)$ for some nonnegative continuous function $\rho$ that depends only on $A$ and $B$ and is defined on the nonnegative reals such that $\rho(0) = 0$.

## III. $S_\beta$ AND THE MAIN THEOREM

In the following definition and hypothesis, $\beta$ is a nonnegative number.

*Definition of* $S_\beta$: $S_\beta$ is the set of operators $H$ in $L^n$ with the following property: For each $v \in L^n$ such that $|v| \neq 0$, there is an index $k$ such that $|v_k| \neq 0$ and $(v_k, (Hv)_k) \geq \beta |v_k|^2$.

The definition of $S_\beta$ is related to one of the equivalent definitions of a "*P*-matrix." [9]

*H.1*: *If* $\beta = 0$, there is a positive constant $\delta$ such that

$$|B_j w| \leq \delta |w| \tag{5}$$

for $w \in L$ and all $j$.

Our main result is the following.

*Theorem 1: Let H.1 (and A.1 through A.3) hold. Then (4) is L-stable if* $A \in S_\beta$ *with* $\alpha + \beta > 0$.

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* In order to focus attention only on essentials, we are proceeding with some assumptions that are stronger than necessary. It will become clear that (2) and (3) (and also (5) of Section III) could have been replaced with somewhat weaker inequalities (similar, for example, to some of those used in Section 5.3 of Ref. 8). Similarly, if for example there are positive constants $\alpha_j$ such that $(B_j w, w) \geq \alpha_j |w|^2$ for $w \in L$ and all $j$, and if $A$ in (4) is represented by an $n \times n$ matrix in the sense of Section IV with $[I + a \text{ diag } (\alpha_j)]$ invertible, then it is clear that $x$ satisfies an equation similar to (4) in which each $B_j x_j$ is replaced with $(B_j x_j - \alpha_j x_j)$ and $A$ and $y$ are modified accordingly. Consideration of such a modified equation often enables a useful trade-off to be made between requirements on $A$ and the degree of positiveness of the $B_j$.

*Proof of Theorem 1*: Let *H.1* and *A.1* through *A.3* be satisfied, let $y \in L^n$, let $x \in K^n$ be a solution of (4), and assume that $A \in S_\beta$ with $\alpha + \beta > 0$.

Suppose first that $\beta = 0$, in which case $\alpha > 0$. Let $\tau \geq 0$ be arbitrary. Using (1) and the casuality of the $A_{ij}$, we have

$$P_\tau x_i + P_\tau \sum_j A_{ij} P_\tau B_j x_j = P_\tau y_i \tag{6}$$

and

$$(P_\tau B_i x_i, P_\tau x_i) + \left(P_\tau B_i x_i, P_\tau \sum_j A_{ij} P_\tau B_j x_j\right) = (P_\tau B_i x_i, P_\tau y_i) \tag{7}$$

for $i = 1,2,\ldots,n$. Since $A \in S_\beta$, and $(P_\tau B_i x_i, P_\tau \Sigma_j A_{ij} P_\tau B_j x_j) = (P_\tau B_i x_i, \Sigma_j A_{ij} P_\tau B_j x_j)$ for each $i$, there is an index $k$ such that $(P_\tau B_k x_k, P_\tau \Sigma_j A_{kj} P_\tau B_j x_j) \geq 0$ and hence such that

$$(P_\tau B_k x_k, P_\tau x_k) \leq (P_\tau B_k x_k, P_\tau y_k). \tag{8}$$

By the Schwarz inequality and the fact that $(P_\tau B_k x_k, P_\tau y_k) = (P_\tau B_k P_\tau x_k, P_\tau y_k) = (B_k P_\tau x_k, P_\tau y_k)$, $(P_\tau B_k x_k, P_\tau y_k) \leq |B_k P_\tau x_k| \cdot |P_\tau y_k|$. Therefore, using $(P_\tau B_k x_k, P_\tau x_k) = (B_k P_\tau x_k, P_\tau x_k)$ as well as (2), (5) and (8), we have

$$\alpha |P_\tau x_k|^2 \leq \delta |P_\tau x_k| \cdot |P_\tau y_k| \tag{9}$$

and consequently, with $c = \delta\alpha^{-1}$,

$$|P_\tau x_k| \leq c|P_\tau y_k|.$$

The argument given above shows that $|P_\tau x_k| \leq c|y|$ for some $k$ (which might depend on $x$ and $\tau$). Let $J$ denote any nonempty proper subset of $\{1,2,\ldots,n\}$ with the following property. For $j \in J$, there is a constant $c_j$ that depends only on $c$, $\delta$, and $\gamma$ such that $|P_\tau x_j| \leq c_j |y|$. Using (1),

$$x_i + \sum_{j \notin J} A_{ij} B_j x_j = y_i - \sum_{j \in J} A_{ij} B_j x_j, \quad i \notin J. \tag{10}$$

The left side of (10) is basically the same in form as the left side of (1). With $r$ the number of elements contained in $J$, let the elements of $(\{1,2,\ldots,n\} - J)$ be $j_1, j_2, \ldots, j_{(n-r)}$ ordered so that $j_1 < j_2 < \cdots < j_{(n-r)}$. With respect to that ordering, let the mapping of $K^{(n-r)}$ into itself associated with (10) that corresponds to $A$ be denoted by $A_J$. Since *A.3* holds, each $A_{ij}$ maps the zero element of $L$ into itself, and it is a simple matter to verify that $A_J$ belongs to, so to speak, $S_\beta$ with $n$ replaced with $(n - r)$. Thus, by the argument given above, we find that there is an index $l \notin J$ such that

$$|P_\tau x_l| \leq c \left| P_\tau \left( y_l - \sum_{j \in J} A_{lj} B_j x_j \right) \right|. \tag{11}$$

Using $|P_\tau x_j| \le c_j|y|$ for $j \in J$, as well as (11) and the causality of the $A_{ij}$ and the $B_j$, we have

$$|P_\tau x_l| \le c\left(|y| + \left|\sum_{j\in J} P_\tau A_{lj} B_j x_j\right|\right)$$

$$\le c\left(|y| + \sum_{j\in J} |P_\tau A_{lj} P_\tau B_j P_\tau x_j|\right)$$

$$\le c\left(|y| + \gamma\delta \sum_{j\in J} |P_\tau x_j|\right)$$

$$\le c\left(1 + \gamma\delta \sum_{j\in J} c_j\right)|y|.$$

Let $\omega_1, \omega_2, \ldots, \omega_n$ be defined by $\omega_1 = c$ and

$$\omega_i = c\left(1 + \gamma\delta \sum_{j=1}^{(i-1)} \omega_j\right), \quad i = 2,3,\ldots,n.$$

We have shown that given $x$, $\tau$, and $i$, $|P_\tau x_i| \le d_i|y|$ for some $d_i \in \{\omega_1, \omega_2, \ldots, \omega_n\}$. Since $\omega_n = \max_i \omega_i$, we have

$$\sum_{i=1}^{n} |P_\tau x_i|^2 \le n\omega_n{}^2 |y|^2 \quad \text{for all } \tau \ge 0,$$

which shows that $x \in L^n$ and that $|x|$ is suitably bounded in terms of $|y|$. This completes the proof for the case in which $\beta = 0$.

The proof for the $\beta > 0$ case is similar. Using primarily (7) and the hypothesis that $A \in S_\beta$, we find that

$$\beta |P_\tau B_k x_k|^2 \le |P_\tau B_k x_k| \cdot |P_\tau y_k|$$

for some $k$. Therefore,

$$|P_\tau B_k x_k| \le \beta^{-1}|y| \tag{12}$$

for some $k$. By proceeding essentially as indicated above, we can show that

$$|P_\tau B_i x_i| \le \Omega_n |y| \quad \text{for all } i \text{ and } \tau \ge 0, \tag{13}$$

in which $\Omega_n$ is the number defined by $\Omega_1 = \beta^{-1}$ and

$$\Omega_i = \beta^{-1}\left(1 + \gamma \sum_{j=1}^{(i-1)} \Omega_j\right)$$

for $i = 2,3,\ldots,n$.

From (6) and (13),

$$|P_\tau x_i| \le |P_\tau y_i| + \sum_j |A_{ij} P_\tau B_j x_i|$$

$$\le |y| + \sum_j \gamma\Omega_n |y|$$

$$\le (1 + n\gamma\Omega_n)|y|$$

for each $i$ and all $\tau \geq 0$. Since tbis shows that $x \in L$ and that $|x|$ is bounded as required, our proof is complete.

### 3.1 Comments

To see that $\bigcup_{\eta \geq 0} S_\eta$ contains tbe familiar set of nonnegative operators defined on a space of vector-valued functions, let $H$ be any mapping of $L^n$ into itself such that

$$\sum_{i=1}^{n} (w_i,(Hw)_i) \geq \sigma |w|^2, \quad w \in L^n, \tag{13}$$

in which $\sigma$ is a nonnegative constant. From (13) it is clear that for each $w \in L^n$ with $|w| \neq 0$ there is an index $k$ such that $|w_k| \neq 0$ and $(w_k,(Hw)_k) \geq \beta |w|^2$ in which $\beta = \sigma n^{-1}$. Since $|w|^2 \geq |w_k|^2$, we observe that $H \in S_\beta$.

Theorem 1 is of course a result concerning the $L$-boundedness of solutions of (4).* By modifying the hypotheses and proof of Theorem 1 in a direct manner, an analogous result can be obtained concerning the $L$-continuity of solutions (i.e., concerning the $L$-boundedness of the difference $(x_a - x_b)$ of a solution $x_a$ of (4) that corresponds to $y = y_a$ and a solution $x_b$ that corresponds to $y = y_b$ with $(y_a - y_b) \in L$). With regard to the necessary modifications of the hypotheses concerning $A$, the following definition, in which $\beta \geq 0$, plays a central role.

*Definition of $T_\beta$*: $T_\beta$ is the set of operators $H$ in $L^n$ with the following property: For each $u$ and $v$ in $L^n$ such that $|u - v| \neq 0$, there is an index $k$ such that $|u_k - v_k| \neq 0$ and $(u_k - v_k, (Hu)_k - (Hv)_k) \geq \beta |u_k - v_k|^2$.

In order to be more explicit, let (*A.1′*) denote the assumption tbat $x_a + ABx_a = y_a$ and $x_b + ABx_b = y_b$ in which each $A_{ij}$ and $B_j$ are causal operators in $K$ and $x_a$, $x_b$, $y_a$, and $y_b$ belong to $K$. Let *A.2′* be the hypothesis obtained from *A.2* by replacing "$(B_jw,w) \geq \alpha |w|^2$ for $w \in L$ and all $j$" with "$(B_ju - B_jv, u - v) \geq \alpha |u - v|^2$ for $u$ and $v$ in $L$ and all $j$," and let *A.3′* and *H.1′* be the hypotheses obtained in a similar manner from *A.3* and *H.1*, respectively.

Our $L$-continuity result (whose proof is omitted) is the following.

*Theorem 2: Let H.1′ and A.1′ through A.3′ be satisfied, let $(y_a - y_b) \in L$, and let $A \in T_\beta$ with $\alpha + \beta > 0$. Then $(x_a - x_b) \in L$, and there is a nonnegative continuous function $\rho$ defined on $[0,\infty)$ that depends only on A and B such that $\rho(0) = 0$ and $|x_a - x_b| \leq \rho(|y_a - y_b|)$.*

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* Results along the lines of Theorem 1 for cases in which $B$ is more general than assumed here but both $A$ and $B$ are nonnegative operators are given in Ref. 8, where the stability of interconnected systems in the sense of Section 2.2 is not explicitly discussed. A nonnegative-operator approach to the stability of interconnected systems, as well as its relation to other approaches, is discussed in Ref. 7.

### 3.2 A Corollary to Theorem 1

We shall refer to the following two hypotheses.

*H.2*: Each $B_i$ is a continuous mapping in $L$ that maps the zero element into itself, and there are positive constants $c_1$ and $c_2$ such that for all $i$

$$(B_i u, u) \geq c_1 |B_i u|^2 \tag{14}$$

$$|B_i u - B_i v| \leq c_2 |u - v| \tag{15}$$

for $u$ and $v$ in $L$.

*H.3*: $K = E_2$, $L = L_2$, and for each $\tau \geq 0$ $P_\tau$ is the operator associated with $E_2$ in the example given in Section 2.1.

There are many cases in which (14) holds* for all $u$ for some $c_1 > 0$, but there is no positive $\alpha$ such that (2) is satisfied for all $w$.† On the other hand, it is clear that there is a positive $c_1$ with the property that (14) is met when (2) holds with $\alpha > 0$ for all $w$ and there is a positive constant $\delta$ such that $|B_i w| \leq \delta |w|$ for all $w$.

*Corollary 1: Let H.2 and H.3 (as well as A.1 and A.3) be satisfied. If $A \in S_0$, then (4) is L-stable.*

*Proof*: Assume that the hypotheses of the corollary are satisfied and let $I$ and $I_n$, respectively, denote the identity operators in $K$ and $K^n$. With regard to the following lemma, two elements $u$ and $v$ of $E_2$ are taken to be the same if and only if $|P_\tau(u - v)| = 0$ for all $\tau \geq 0$.

*Lemma 1: Let H.3 hold, let $F$ be a continuous mapping of $L_2$ into itself such that for some positive constant $c < 1$ we have*

$$|Fu - Fv| \leq c|u - v| \quad \text{for } u \text{ and } v \text{ in } L_2, \tag{16}$$

*and let $F$ also be a causal mapping of $E_2$ into $E_2$. Then $(I - F)^{-1}$ exists and is causal on both $L_2$ and $E_2$.*

*Proof of Lemma 1*: Let the hypotheses of the lemma be met. In view of (16) and the continuity of $F$, the equation $x - Fx = h$ with $h \in L_2$ has in $L_2$ a unique solution $x$ which is given by $x = \lim_{n\to\infty} x^{(n)}$ in which $x^{(n)} = h + Fx^{n-1}$ for $n \geq 1$ and $x^{(0)} = h$. Thus, $(I - F)^{-1}$ exists on $L_2$, and since $h + F(\cdot)$ is causal on $L_2$ so is $(I - F)^{-1}$.

Now let $h \in E_2$, and for each $\tau \geq 0$ let $z_\tau$ be the unique element of $L_2$ that satisfies $z_\tau - Fz_\tau = P_\tau h$. Since $(I - F)^{-1}$ is causal on $L_2$, it is clear that $P_{\tau_1} z_{\tau_2} = P_{\tau_1} z_{\tau_1}$ for $\tau_2 \geq \tau_1$. Let $x$ be the element of $E_2$ defined by the condition that $P_\tau x = P_\tau z_\tau$ for all $\tau \geq 0$. For any $\tau \geq 0$, $P_\tau x - P_\tau Fx = P_\tau z_\tau - P_\tau FP_\tau x = P_\tau z_\tau - P_\tau FP_\tau z_\tau = P_\tau z_\tau - P_\tau Fz_\tau = P_\tau h$. Therefore $x$ satisfies $x - Fx = h$. Suppose that $x_1$ and $x_2$ in $E_2$ satisfy $h = x_1 - Fx_1 = x_2 - Fx_2$

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* This type of inequality is among those used in Ref. 8.

† We mention two simple examples: Let $L = L_2$ and let $B_i$ be defined by the condition that for each $t \geq 0$, $(B_i w)(t) = w(t)$ for $|w(t)| \leq 1$ and $(B_i w)(t) = \operatorname{sgn}(w(t))$ for $|w(t)| > 1$. Then (14) with $c_1 = 1$ holds for all $u \in L_2$, but there is no $\alpha > 0$ for which (2) is satisfied for all $w \in L_2$. It is not difficult to show that a similar conclusion is reached when $B_i$ is the convolution operator in $L_2$ with impulse response $e^{-t}$.

with $|P_\tau(x_1 - x_2)| \neq 0$ for some $\tau \geq 0$. Since $F$ is causal, we have $P_\tau h_1 = P_\tau h_2$ in which $h_1 = P_\tau x_1 - FP_\tau x_1$ and $h_2 = P_\tau x_2 - FP_\tau x_2$. This contradicts the fact that $(I - F)^{-1}$ is causal on $L_2$. Thus, $x$ is the unique solution in $E_2$ of $x - Fx = h$, which means that $(I - F)^{-1}$ exists on $E_2$, because $h$ is arbitrary. In view of the fact that the solution $x$ of $x - Fx = h$ satisfies $P_\tau x = P_\tau z_\tau$ where $z_\tau - Fz_\tau = P_\tau h$ for every $\tau \geq 0$, it is evident that the operator $(I - F)^{-1}$ on $E_2$ is causal. This proves the lemma.

Let $c$ be a positive constant such that $c < \min(c_1, c_2^{-1})$. By Lemma 1, $(I_n - cB)^{-1}$ exists on $K^n$ and $B(I_n - cB)^{-1}$ is causal on $K^n$ and maps $L^n$ into itself. In particular, the equation $x + ABx = y$ can be written as

$$h + (A + cI_n)B(I_n - cB)^{-1}h = y$$

in which $h = x - cBx$. From $A \in S_0$, it follows at once that $(A + cI_n) \in S_c$.

Also, from $h = x - cBx$ and the fact that $B$ is causal on $K^n$ and satisfies $|Bu| \leq c_2|u|$ for $u \in L^n$, with $cc_2 < 1$, we have $|P_\tau x| \leq (1 - cc_2)^{-1}|P_\tau h|$ for $\tau \geq 0$.

Therefore, by Theorem 1, to complete the proof it suffices to observe that for any $w \in L^n$,

$$\begin{aligned}(B(I_n - cB)^{-1}w, w) &= (Bu, u - cBu)\\ &= (Bu, u) - c|Bu|^2\\ &\geq 0\end{aligned}$$

in which of course $u = (I_n - cB)^{-1}w$.

## IV. RESULTS CONCERNING THE MATRIX CASE

Of importance in the theory of interconnected systems is the special case in which $A$ is represented by a real $n \times n$ matrix $a$ with elements $a_{ij}$, in the sense that for each $i$,

$$(Aw)_i = \sum_{j=1}^{n} a_{ij}w_j, \quad w \in L^n.$$

Throughout this section, "$A \in M$" means that $A$ has such a representation with representation matrix $a$, and, assuming that $A \in M$, $U_0(U)$ denotes the set of representation matrices such that $A \in S_0$ ($A \in S_\beta$ with $\beta > 0$). In addition, $P_0(P)$ denotes the set of real square matrices with nonnegative (positive) principal minors.

*Proposition 1: If (A.1 through A.3 are satisfied and) $A \in M$ with $a \notin P_0$, then (4) is not L-stable for some B.*

*Proof*: Let the hypotheses be met, and let $1_n$ denote the identity matrix of order $n$. From $a \notin P_0$, it follows that there is a diagonal matrix $d =$

diag $(d_1,d_2,\ldots,d_n)$ with $d_i > 0$ for all $i$ such that $(d + a)$ and hence $(1_n + ad^{-1})$ are singular.*

For each $i$, let $B_i$ be defined by the condition that $B_i w = d_i^{-1} w$ for $w \in K$. Let $v$ be any real nonzero $n$-vector that is annihilated by $(1_n + ad^{-1})$, and let $e$ be any element of $K$ different from the zero element $\theta$ of $L$. With $x = ve$, we have $x + ABx = \theta$. This shows that (4) is not $L$-stable for the particular $B$ constructed.

In order to proceed to the first theorem in this section, it is necessary to introduce the following definitions. For each $w \in L^n$, $a_w$ denotes the matrix obtained from $a$ by replacing $a_{ij}$ with $a_{ij}(w_i,w_j)$ for all $i$ and $j$. For any $w \in L^n$ with $|w| \neq 0$, $\Gamma(a_w)$ denotes the matrix obtained from $a_w$ by deleting both the $i$th row and $i$th column for all $i \in \{i: |w_i| = 0\}$.

*Theorem 3: Let $A \in M$. We have $a \in U(a \in U_0)$ if and only if $\Gamma(a_w) \in P(a_w \in P_0)$ for each $w \in L^n$ with $|w| \neq 0$.*

*Proof of Theorem 3*: We shall use two lemmas. With regard to the first of the lemmas, $M_n$ denotes the normed linear space of real $n \times n$ matrices, with the usual Euclidean norm, and $C$ denotes $\{u \in M_n$: there is a $w \in L^n$ with $|w| = 1$ such that $u_{ij} = (w_i,w_j)$ for all $i$ and $j\}$.

*Lemma 2: C is compact.*

*Proof of Lemma 2*: The set $C$ is obviously bounded. To show that $C$ is closed, let $u^{(1)},u^{(2)},\ldots$ be a sequence of elements of $C$ that converges to some element $\bar{u}$ of $M_n$.

Given a real $n$-vector $v$, for each $w \in L^n$ and its corresponding element $u$ of $C$, we have $v^{tr}uv = (\Sigma_i v_i w_i, \Sigma_i v_i w_i) \geq 0$.† Thus, each $u^{(j)}$ is nonnegative definite, and therefore it follows that $\bar{u}$ is nonnegative definite. In view of the fact that $Tr(u^{(j)}) = 1$ for all $j$, it also follows that $Tr(\bar{u}) = 1$.

Since $\bar{u}$ is nonnegative definite and has unit trace, there is an orthogonal matrix $T$ and a diagonal matrix $D = \text{diag}\,(d_1,d_2,\ldots,d_n)$ with $d_j \geq 0$ and $\Sigma_j d_j = 1$ such that

$$\bar{u} = TDT^{tr} = \sum_l d_l T_l (T_l)^{tr},$$

in which $T_l$ is the $l$th column of $T$. Referring to the pairwise mutually orthogonal elements $e_1,e_2,\ldots,e_n$ mentioned in Section 2.1, let $z$ in $L^n$ be defined by

$$z = \sum_i d_i^{1/2} T_i e_i.$$

Using the orthonormality of the $e_i$, and the fact that for each $l$ the sum of the squares of the components of $T_l$ is unity, it is not difficult to verify that $|z| = 1$, and that $(z_i,z_j)$ is equal to the $i,j$th element of $\bar{u}$ for all $i$ and

* A proof is given in Ref. 10. Another proof can be obtained from the fact that, since $a \notin P_0$, there is (see Ref. 9) a real nonzero $n$-vector $q$ such that $q_i(aq)_i < 0$ for every $i$ such that $q_i \neq 0$.

† The superscript "$tr$" denotes transpose.

$j$. This shows that $C$ is closed, and completes the proof of the lemma.*

The following lemma is proved in Ref. 9.

*Lemma 3: A real square matrix $m$ belongs to $P(P_0)$ if and only if $v_k(mv)_k > 0$ ($v_k \neq 0$ and $v_k(mv)_k \geq 0$) for some $k$ for each real nonzero vector $v$ of dimension equal to the order of $m$.*

In order to prove the theorem, suppose initially that $\Gamma(a_w) \in P$ for every $w \in L^n$ with $|w| \neq 0$. By Lemma 3, for each $w \in L^n$ with $|w| = 1$ we have $(\Gamma(a_w)v)_k > 0$ for some index $k$, when all of the components of the vector $v$ of compatible dimension are unity. Thus, $\max_i \Sigma_{j=1}^n a_{ij}(w_i,w_j)$, which we view as a function of the matrix $u$ whose elements are the $(w_i,w_j)$, is positive for each $w$ in $L^n$ with unit norm. Since $\max_i \Sigma_{j=1}^n a_{ij}(w_i,w_j)$ is obviously a continuous function of $u$, and, by Lemma 2, $C$ is compact, there is a $\sigma > 0$ such that

$$\min_{u\in C} \max_{i} \sum_{j=1}^{n} a_{ij}(w_i,w_j) = \sigma. \tag{17}$$

Therefore, for each $w \in L^n$ with $|w| = 1$ there is an index $k$ such that

$$\sum_{j=1}^{n} a_{kj}(w_k,w_j) \geq \sigma|w|^2 \geq \sigma|w_k|^2,$$

from which we see that for each $w \in L^n$ with $|w| \neq 0$, there is a $k$ such that $|w_k| \neq 0$ and

$$\sum_{j=1}^{n} a_{kj}(w_k,w_j) \geq \sigma|w_k|^2.$$

Thus, $a \in U$.

To show that $a \in U_0$ when $a_w \in P_0$ (and hence $\Gamma(a_w) \in P_0$) for each $w \in L^n$ with $|w| > 0$, we observe tbat then, by Lemma 3, for each $w \in L^n$ with $|w| \neq 0$ we have $(\Gamma(a_w)v)_k \geq 0$ for some $k$ when the components of $v$ are all unity. Therefore, for each $w \in L^n$ with $|w| \neq 0$, there is a $k$ such that $|w_k| \neq 0$ and

$$\sum_{j=1}^{n} a_{kj}(w_k,w_j) \geq 0,$$

which means that $a \in U_0$.

Suppose now that for some $w \in L^n$ with $|w| > 0$ we have $\Gamma(a_w) \notin P(a_w \notin P_0)$. Then, by Lemma 3, there is a nonzero vector $v$ such that $v_k(\Gamma(a_w)v)_k \leq 0$ ($v_k(\Gamma(a_w)v)_k < 0$) for every $k$ such that $v_k \neq 0$. Thus, by multiplying each $w_i$ for which $|w_i| \neq 0$ by the appropriate component of $v$, it is a simple matter to construct a $z \in L^n$ for which $|z| \neq 0$ and $\Sigma_j a_{ij}(z_i,z_j) \leq 0$ for all $i(\Sigma_j a_{ij}(z_i,z_j) < 0$ for all $i$ such that $|z_i| \neq 0)$. This completes the proof of the theorem.

*Corollary 2: If $n \geq 3$, $U(U_0)$ is a proper subset of the matrices of order $n$ in $P(P_0)$.*

* Of some peripheral interest is the fact that it is not necessary to assume that $L$ is complete.

*Proof*: To see that $U(U_0)$ is a subset of $P(P_0)$, let $a \in U(U_0)$, let $e$ be any element of $L$ such that $|e| = 1$, and let $w$ be the element of $L^n$ defined by $w_i = e$ for all $i$. Thus $a_w = a$, and, by Theorem 3, $a \in P(P_0)$.

In order to show that for $n \geq 3$ there is a matrix of order $n$ in $P(P_0)$ that is not contained in $U(U_0)$, observe that it is sufficient to consider the $n = 3$ case, and let $a^{(+)}$ and $a^{(0)}$ be defined by

$$a^{(+)} = \begin{bmatrix} 1.1 & 1 & -10 \\ 1 & 1.1 & 1 \\ 1 & 1 & 1.1 \end{bmatrix},$$

$$a^{(0)} = \begin{bmatrix} 1 & 1 & -10 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

We have $a^{(+)} \in P$ and $a^{(0)} \in P_0$. Let $w \in L^3$ be given by

$$w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e_1 + \begin{pmatrix} 0 \\ 1 \\ 10 \end{pmatrix} e_2,$$

in which $e_1$ and $e_2$ are orthogonal elements of $L$ with unit norm. It is a simple matter to verify that

$$a_w^{(+)} = \begin{bmatrix} 1.1 & 1 & -10 \\ 1 & 2.2 & 11 \\ 1 & 11 & 111.1 \end{bmatrix},$$

$$a_w^{(0)} = \begin{bmatrix} 1 & 1 & -10 \\ 1 & 2 & 11 \\ 1 & 11 & 101 \end{bmatrix}$$

and that we have $\det[a_w^{(+)}] < 0$ and $\det[a_w^{(0)}] < 0$. By Theorem 3, this shows that $a^{(+)}(a^{(0)}) \notin U(U_0)$, which completes the proof.

*Corollary 3: If $n = 2$, $U(U_0) = P(P_0)$ restricted to 2×2 matrices.*

This follows directly from Theorem 3.*

### 4.1 Definitions

Let $N$ denote the set of real symmetric nonnegative definite matrices $m$ of order $n$ with $m_{ii} > 0$ for some $i$. For each $m \in N$, let $a_m$ denote the $n \times n$ matrix whose $i,j$th element is $a_{ij}m_{ij}$ for all $i$ and $j$,† and let $\Lambda(a_m)$ denote the matrix obtained from $a_m$ by deleting row $i$ and column $i$ for each $i \in \{j: m_{jj} = 0\}$.

*Corollary 4*: We have $a \in U(U_0)$ if and only if $\Lambda(a_m) \in P(a_m \in P_0)$ for each $m \in N$.

*Proof*: The proof of Lemma 2‡ shows that a real matrix $m$ of order $n$ belongs to $N$ if and only if there is a $w$ in $L^n$ such that $|w| \neq 0$ and $m_{ij} = (w_i,w_j)$ for all $i$ and $j$. Thus, Corollary 4 follows from Theorem 3.

### 4.2 Introduction to Corollary 5

In order to present our next corollary, we need the following additional definitions: Let $S(m)$ denote the set of all matrices obtainable from a given real $n \times n$ matrix $m$ by replacing each *off-diagonal* element $m_{ij}$ of $m$ with $r_{ij}m_{ij}$, where the $r_{ij}$ are real numbers that satisfy $r_{ij} = r_{ji}$ and $|r_{ij}| \leq 1$. Let $R$ denote $\{m \in P: S(m) \subset P\}$, and, similarly, let $R_0 = \{m \in P_0: S(m) \subset P_0\}$.

When $n = 2$ and $P$ and $P_0$ are restricted to 2×2 matrices, we have $R = P$ and $R_0 = P_0$. On the other hand, if we let $a^{(+)}(\lambda)$ and $a^{(0)}(\lambda)$, respectively, denote the matrices obtained from $a^{(+)}$ and $a^{(0)}$ of the proof of Corollary 2 by multiplying the (1,3) and (3,1) elements by a scalar variable $\lambda$, then $a^{(+)}(1) \in P$ and $a^{(0)}(1) \in P_0$, but $a^{(+)}(0) \notin P$, and, similarly, $a^{(0)}(0) \notin P_0$. This shows that $R(R_0)$ is a proper subset of the $n \times n$ matrices in $P(P_0)$ when $n \geq 3$.§ Two familiar classes of matrices contained, for example, in $R_0$ are the set of row-sum dominant matrices and the set of column-sum dominant matrices.

*Corollary 5: We have $a \in U(U_0)$ if either*

*(i) $a \in R(R_0)$.*

*(ii) There are diagonal matrices $d_1$ and $d_2$ of order $n$ with positive diagonal elements such that $d_1 a d_2$ is positive definite (nonnegative definite).*¶

*Proof*: Suppose first that $a \in R(R_0)$. Let $w$ be any element of $L^n$ such that $|w| \neq 0$, let $c = \mathrm{diag}(c_1,c_2, \ldots ,c_n)$ in which for all $i$, $c_i = 0$ if $|w_i|$

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* The proof of Corollary 2 shows that $U \subset P$ and $U_0 \subset P_0$ for $n \geq 2$.

† In other words, let $a_m$ denote the "Schur product" of $a$ and $m$.

‡ Lemma 2 is used in the proof of Theorem 3.

§ It will become clear that this proposition also follows from Corollary 2 and Corollary 5.

¶ As usual, we say that a real square matrix $m$ is positive definite (nonnegative definite) if and only if the symmetric part of $m$ is the matrix of a positive definite (nonnegative definite) quadratic form.

$= 0$ and $c_i = |w_i|^{-1}$ if $|w_i| > 0$, and let $\Gamma(ca_w c)$ denote the matrix obtained from $(ca_w c)$ by deleting the rows and columns corresponding to the indices $i$ for which $c_i = 0$. In view of the fact that $|(w_i,w_j)| \leq |w_i|\cdot|w_j|$ for each $i$ and $j$, we see that $\Gamma(ca_w c) \in P(P_0)$ and hence* that $\Gamma(a_w) \in P(P_0)$. By Theorem 3, $a \in U(U_0)$.

At this point, we need the following lemma.

*Lemma 4*†: *If $p = \{p_{ij}\}$ and $q = \{q_{ij}\}$ are real square matrices of the same order, with $p$ positive definite and $q$ symmetric, nonnegative definite, and such that $q_{ii} > 0$ for all $i$, then $r = \{p_{ij}q_{ij}\}$ is positive definite.*

*Proof of Lemma 4*: Let $p$ and $q$ be as indicated, and let $k$ denote the order of $p$. The proof of Lemma 2 shows that $L_2$ contains $k$ functions $f_1, f_2, \ldots, f_k$ such that

$$q_{ij} = \int_0^\infty f_i(t)f_j(t)dt \quad \text{for all } i \text{ and } j.$$

With $v$ any real nonzero $k$-vector and with $\lambda$ the smallest eigenvalue of the symmetric part of $p$, we have

$$\begin{aligned} v^{tr}rv &= \sum_{i,j} v_i v_j p_{ij} \int_0^\infty f_i(t)f_j(t)dt \\ &= \int_0^\infty \sum_{i,j} p_{ij} v_i f_i(t) v_j f_j(t) dt \\ &\geq \int_0^\infty \lambda \sum_i (v_i f_i(t))^2 dt \\ &> 0, \end{aligned}$$

which shows that $r$ is positive definite.

To complete the proof of the corollary, suppose that $d_1 a d_2$ is positive definite, with $d_1$ and $d_2$ as described, and let $m \in N$. By Lemma 4, $\Lambda\{d_1 a_m d_2\}$ (i.e., $\Lambda\{a_m\}$ with $a$ replaced with $d_1 a d_2$) is positive definite and hence it belongs to $P$. Therefore, $\Lambda\{a_m\} \in P$, and, by Corollary 4, $a \in U$.

The proof for the case in which $d_1 a d_2$ is nonnegative definite is essentially the same, and is omitted.

### 4.3 Comments Regarding Corollary 5

In light of the fact that $R_0 = P_0$ restricted to 2×2 matrices when $n = 2$, the following special result is a direct consequence of Corollaries 5 and 1, and the content of the proof of Proposition 1.

*Proposition 2: Let $n = 2$ and $A \in M$. Let H.2 and H.3 (as well as A.1 and*

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* Here and in another part of the proof, we use the easily proved result that a real square matrix $m$ belongs to $P(P_0)$ if and only if $d_1 m d_2 \in P(P_0)$ for every pair of compatible diagonal matrices $d_1$ and $d_2$ with positive diagonal elements.

† A proof that the conclusion of Lemma 4 holds when $q$ is positive definite is given in Ref. 11.

*A.3) be satisfied. Then (4) is L-stable for every B if and only if* $a \in P_0$.

An example of a matrix $a$ that is nonnegative definite and such that $a \notin R_0$ is given by

$$a = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2^{1/2} & 2^{1/2} \\ 1 & 2^{1/2} & 2^{1/2} \end{bmatrix},$$

since

$$\det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2^{1/2} & 2^{1/2} \\ 0 & 2^{1/2} & 2^{1/2} \end{bmatrix} < 0.$$

Similarly, a very simple example of an $a \in R_0$ such that $d_1 a d_2$ is nonnegative definite for no suitable $d_1$ and $d_2$ is

$$a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Of some interest is the fact that $a \in U$ if $a$ is an $M$-matrix (i.e., if $a$ has positive principal minors and nonpositive off-diagonal elements); in that case there is a diagonal matrix $d$ with positive diagonal elements such that $ad$ is strongly row-sum dominant* and therefore $ad$ and consequently $a$ belong to $R$.

*Theorem 4: Let* $A \in M$. *If* $a_{ii} = 0$ *for all i, then* $a \in U_0$ *if and only if* $a \in R_0$.

*Proof*: The "if part" is a special case of Corollary 5.

Suppose that $a \in U_0$ with $a_{ii} = 0$ for all $i$, and suppose also that $a \notin R_0$ in which case there is an element $b$ of $S(a)$ such that $b \notin P_0$. Let $b$ be given by $b_{ii} = 0$ for all $i$, and $b_{ij} = r_{ij} a_{ij}$ with $r_{ij} = r_{ji}$ for $i \neq j$. Choose $n$ real numbers $r_{11}, r_{22}, \ldots, r_{nn}$ so that the $n \times n$ matrix $m$ given by $m_{ij} = r_{ij}$ for all $i$ and $j$ is nonnegative definite. Observe that $m \in N$. Since $a_m = b$, by Corollary 4, we have a contradiction to the supposition that $a \in U_0$. Therefore, $a \in R_0$ when $a \in U_0$ and $a_{ii} = 0$ for all $i$, which completes the proof of the theorem.

### 4.4 Comment regarding Theorem 4

We can have $a \in P_0$ with $a_{ii} = 0$ for all $i$, and $a \notin R_0$. For example, let

$$a = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

* See the theorem given on page 387 of Ref. 12.

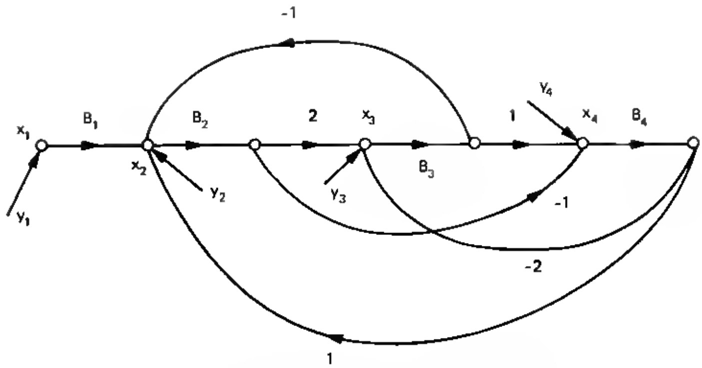


Fig. 1—Flow graph of an interconnected system.

Then $a \in P_0$, and

$$\det \begin{bmatrix} 0 & -r_{12} & r_{13} \\ r_{12} & 0 & 0 \\ 0 & r_{23} & 0 \end{bmatrix} = r_{23} r_{12} r_{13} < 0$$

for, say, $r_{23} = r_{12} = -r_{13} = 1$.

### *4.5 A Specific example of an L-stable interconnected system*

Assume that *H.2* and *H.3* (as well as *A.1* and *A.3*) are satisfied. For the system described in flow-graph form in Fig. 1, we have

$$y_1 = x_1$$
$$y_2 = x_2 - B_1 x_1 + B_3 x_3 - B_4 x_4$$
$$y_3 = x_3 - 2B_2 x_2 + 2B_4 x_4$$
$$y_4 = x_4 + B_2 x_2 - B_3 x_3.$$

Here $A \in M$, with

$$a = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 \\ 0 & -2 & 0 & 2 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$

To see that $a \in R_0$, consider the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -r_{21} & 0 & r_{23} & -r_{24} \\ 0 & -2r_{23} & 0 & 2r_{34} \\ 0 & r_{24} & -r_{34} & 0 \end{bmatrix}.$$

We observe that its determinant vanishes and every 1×1 and 2×2 principal minor is nonnegative for all real values of the $r_{ij}$. It is a simple matter to verify that its principal minor of order three obtained by deleting the first row and first column vanishes for all values of the $r_{ij}$, and it is clear that every other principal minor of order three also vanishes for all values of the $r_{ij}$.

Since $a \in R_0$, by Corollary 1 and either Corollary 5 or Theorem 4, the system described in Fig. 1 is $L$-stable.

Another way to prove that the system in Fig. 1 is $L$-stable is as follows. Since $H.2$ holds, $|B_1 u| \leq c_2 |u|$ for $u \in L$. It therefore suffices to show the $L$-stability of the system obtained from the flow graph in Fig. 1 by deleting $B_1$, $x_1$, and $y_1$. That can be done with the aid of Corollary 5 by verifying that the interconnection matrix $a$ of the modified system has the property that there is a 3×3 diagonal matrix $d$ with positive diagonal elements such that $da$ is nonnegative definite.

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